

RANK-BASED SLOCC CLASSIFICATION FOR ODD N QUBITS

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Abstract

We study the entanglement classification under stochastic local operations and classical communication (SLOCC) for odd n -qubit pure states. For this purpose, we introduce the rank with respect to qubit i for an odd n -qubit state. The ranks with respect to qubits $1, 2, \dots, n$ give rise to the classification of the space of odd n qubits into 3^n families.

keywords: Entanglement classification, SLOCC equations, odd n qubits

1 Introduction

Quantum entanglement plays a crucial role in quantum computation and quantum information processing. If two states can be obtained from each other by means of local operations and classical communication with nonzero probability (SLOCC), then the two states are said to have the same kind of entanglement [1] and suited to do the same tasks of quantum information theory [2].

The complete classification for three qubit pure states has been achieved [2]. While there are six SLOCC equivalence classes for pure states of three qubits, two of which are genuine entanglement classes: the $|GHZ\rangle$ class and the $|W\rangle$ class, the number of SLOCC equivalent classes for four or more qubits is infinite. An important first step in tackling the classification problem for four or more qubits is to divide the infinite SLOCC classes into a finite number of families, using some type of criteria to determine which family an arbitrary state belongs to. Many efforts have been devoted to the SLOCC entanglement classifications for pure states of four qubits which result in different finite number of families or classes, including those based on Lie group theory [3], on hyperdeterminant [4], on inductive approach [5], and on string theory [6]. Polynomial invariants for four and five qubits [7, 8, 9] as well as for n qubits [10, 11] have been discussed, and several attempts have been made for SLOCC classification via the vanishing or not of the polynomial invariants [11, 12, 13, 14, 15, 16]. Recently, entanglement classification for the symmetric n -qubit states has been achieved by introducing two parameters called the diversity degree and the degeneracy configuration [17].

In this paper, we investigate SLOCC classification of odd n -qubit pure states. To this end, we introduce the rank with respect to qubit i for an odd n -qubit state and establish its invariance under SLOCC. The rank with respect to qubit i ranges over the values 0, 1, 2, and therefore gives rise to the classification of the space of odd n qubits into 3 families, as exemplified here. Furthermore, the ranks with respect to qubits $1, 2, \dots, n$, permit the partitioning of the space of the pure states of odd $n \geq 5$ qubits into 3^n inequivalent families under SLOCC. We also characterize pure biseparable states and genuinely entangled states in terms of the ranks.

The paper is organized as follows. In section 2, we introduce the rank with respect to qubit i for any state of odd $n \geq 3$ qubits. In section 3, we investigate SLOCC classification of odd n qubits. We give the brief discussion in section 4 and the conclusion in section 5.

2 Rank of a state with respect to qubit i

For odd n qubits, let the state $|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle$, where $|i\rangle$ are basis states and a_i are coefficients. Let the 2×2 matrix

$$M(|\psi\rangle) = \begin{pmatrix} P(|\psi\rangle) & T(|\psi\rangle) \\ T(|\psi\rangle) & Q(|\psi\rangle) \end{pmatrix}, \quad (2.1)$$

where $T(|\psi\rangle)$, $P(|\psi\rangle)$ and $Q(|\psi\rangle)$ are three quantities defined on the space of pure states of odd n qubits:

$$T(|\psi\rangle) = \sum_{i=0}^{2^{n-1}-1} (-1)^{N(i)} a_i a_{2^n-i-1}, \quad (2.2)$$

$$P(|\psi\rangle) = 2 \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} a_{2i} a_{2^{n-1}-2i-1}, \quad (2.3)$$

$$Q(|\psi\rangle) = 2 \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} a_{2^{n-1}+2i} a_{2^{n-2}-2i-1}. \quad (2.4)$$

Here $N(i)$ is the parity of i (i.e. the number of 1's in the binary representation of i). Clearly $M(|\psi\rangle)$ is symmetric and the rank of $M(|\psi\rangle)$ ranges over the values 0, 1, 2. We refer to the rank of $M(|\psi\rangle)$ as the rank of the state $|\psi\rangle$ with respect to qubit 1.

As the quantities $T(|\psi\rangle)$, $P(|\psi\rangle)$ and $Q(|\psi\rangle)$ vary under transpositions $(1, i)$ on qubits 1 and i ($2 \leq i \leq n$), so in general does the rank of the state $|\psi\rangle$ with respect to qubit 1. The variance allows one to define the rank of a state with respect to qubit i ($2 \leq i \leq n$). For this purpose, we first let $T^{(i)}(|\psi\rangle)$, $P^{(i)}(|\psi\rangle)$ and $Q^{(i)}(|\psi\rangle)$ be obtained from $T(|\psi\rangle)$, $P(|\psi\rangle)$ and $Q(|\psi\rangle)$, respectively, under transposition $(1, i)$ on qubits 1 and i , namely

$$T^{(i)}(|\psi\rangle) = (1, i)T(|\psi\rangle), \quad (2.5)$$

$$P^{(i)}(|\psi\rangle) = (1, i)P(|\psi\rangle), \quad (2.6)$$

$$Q^{(i)}(|\psi\rangle) = (1, i)Q(|\psi\rangle), \quad (2.7)$$

for $i = 1, 2, \dots, n$. It is trivial to see that $T^{(1)}(|\psi\rangle) = T(|\psi\rangle)$, $P^{(1)}(|\psi\rangle) = P(|\psi\rangle)$, and $Q^{(1)}(|\psi\rangle) = Q(|\psi\rangle)$.

Analogously, we can construct $M^{(i)}(|\psi\rangle)$ as

$$M^{(i)}(|\psi\rangle) = \begin{pmatrix} P^{(i)}(|\psi\rangle) & T^{(i)}(|\psi\rangle) \\ T^{(i)}(|\psi\rangle) & Q^{(i)}(|\psi\rangle) \end{pmatrix}. \quad (2.8)$$

Note that $M^{(i)}(|\psi\rangle)$ can also be obtained from $M(|\psi\rangle)$ by taking transpositions $(1, i)$ on qubits 1 and i . Clearly, $M^{(i)}(|\psi\rangle)$ is a symmetric matrix and $M^{(1)}(|\psi\rangle) = M(|\psi\rangle)$. The rank of the matrix $M^{(i)}(|\psi\rangle)$ in Eq. (2.8) is referred to as the rank of the state $|\psi\rangle$ with respect to qubit i and denoted as $rank^{(i)}(|\psi\rangle)$.

For example, for three qubits, we obtain $rank^{(i)}(|W\rangle) = 1$ for $i = 1, 2, 3$, whereas for any odd n qubits, we find that $rank^{(i)}(|GHZ\rangle) = 2$ and $rank^{(i)}(|0 \dots 0\rangle) = 0$ for $i = 1, \dots, n$.

Next, we establish the invariance of the rank for any state of odd n qubits under SLOCC. Let $|\psi'\rangle$ be another odd n -qubit state with $|\psi'\rangle = \sum_{i=0}^{2^n-1} b_i |i\rangle$. Recall that if two states $|\psi\rangle$ and $|\psi'\rangle$ are SLOCC equivalent, then there exist invertible local operators $\mathcal{A}_1, \mathcal{A}_2, \dots$ and \mathcal{A}_n ($\det(\mathcal{A}_i) \neq 0$) such that [2]

$$|\psi\rangle = \underbrace{\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \dots \otimes \mathcal{A}_n}_n |\psi'\rangle. \quad (2.9)$$

Then, we assert that if $|\psi\rangle$ and $|\psi'\rangle$ are SLOCC equivalent then the following SLOCC matrix equation holds (see Appendix A for the proof):

$$M^{(i)}(|\psi\rangle) = \mathcal{A}_i M^{(i)}(|\psi'\rangle) \mathcal{A}_i^T \det(\mathcal{A}_1) \dots \det(\mathcal{A}_{i-1}) \det(\mathcal{A}_{i+1}) \dots \det(\mathcal{A}_n), \quad (2.10)$$

where $M^{(i)}(|\psi'\rangle)$ is obtained from $M^{(i)}(|\psi\rangle)$ by replacing $|\psi\rangle$ by $|\psi'\rangle$.

It follows from Eq. (2.10) that the rank of the matrix $M^{(i)}(|\psi\rangle)$ in Eq. (2.8) is invariant under SLOCC, thereby revealing that the rank of the state $|\psi\rangle$ with respect to qubit i is an inherent property. Then the following result holds: if two states are SLOCC equivalent, then they have the same rank with respect to the same qubit i . It should be noted that the converse does not hold, i.e., two states with the same rank with respect to the same qubit are not necessarily equivalent.

To exemplify, we consider the n -qubit symmetric Dicke states $|\ell, n\rangle$ with ℓ excitations, $1 \leq \ell \leq (n-1)$ [18]:

$$|\ell, n\rangle = \binom{n}{\ell}^{-1/2} \sum_k P_k |1_1, 1_2, \dots, 1_\ell, 0_{\ell+1}, \dots, 0_n\rangle, \quad (2.11)$$

where $\{P_k\}$ is the set of all distinct permutations of the spins. For any odd $n \geq 3$ qubits, a straightforward calculation yields $\text{rank}^{(i)}(|(n-1)/2, n\rangle) = 1$, $i = 1, \dots, n$. For any odd $n \geq 5$ qubits, $\text{rank}^{(i)}(|\ell, n\rangle) = 0$ (note that $\text{rank}^{(i)}(|W\rangle) = 0$ as well, since n -qubit $|W\rangle$ state is identical with $|1, n\rangle$ for $1 \leq \ell < (n-1)/2$ and $i = 1, \dots, n$. Since the Dicke states $|\ell, n\rangle$ and $|(n-\ell), n\rangle$ are SLOCC equivalent, the rank for any Dicke state can be determined.

Now consider pure biseparable states, i.e., those that are separable under some bipartition. By virtue of Theorem 3.4 of [19], we arrive at a necessary condition for a pure state to be biseparable: if $|\psi\rangle$ is a pure biseparable state of odd n qubits, then $\text{rank}^{(i)}(|\psi\rangle) = 0$ or 1 for some i with $1 \leq i \leq n$.

In view of the above condition and the fact that a pure state of n qubits is genuinely entangled if it is not biseparable, we obtain the following sufficient condition for a pure state to be genuinely entangled: for any pure state $|\psi\rangle$ of odd n qubits, if $\text{rank}^{(i)}(|\psi\rangle) = 2$ for any $1 \leq i \leq n$, then $|\psi\rangle$ is genuinely entangled.

Remark. If we take the absolute value of the determinant of $M^{(i)}(|\psi\rangle)$ given in Eq. (2.8), then we obtain the n -tangle with respect to qubit i of odd n qubits $\tau_{12\dots n}^{(i)}$ given in [20] (up to a constant factor). In particular, when $n = 3$, $|\det M^{(i)}(|\psi\rangle)|$ is, up to a constant factor, equal to the 3-tangle [21] (we refer the reader to [10] for more details). Further, taking the determinants of both sides of Eq. (2.10) yields

$$\det M^{(i)}(|\psi\rangle) = \det M^{(i)}(|\psi'\rangle) [\det(\mathcal{A}_1) \cdots \det(\mathcal{A}_n)]^2. \quad (2.12)$$

Note that for $i = 1$, we recover Eq. (2.16) of [10]. It follows from Eq. (2.12) that if one of $\det M^{(i)}(|\psi\rangle)$ and $\det M^{(i)}(|\psi'\rangle)$ vanishes while the other does not, then the state $|\psi\rangle$ is not equivalent to $|\psi'\rangle$ under SLOCC. Clearly, the SLOCC invariance of the rank of $M^{(i)}(|\psi\rangle)$ is stronger than the invariance of the determinant.

3 SLOCC classification of odd n qubits

3.1 Three families based on the rank with respect to qubit i

The rank with respect to qubit i permits the partitioning of the space of the pure states of odd n qubits into the following three families: $F_{r_i}^{(i)} = \{|\psi\rangle : \text{rank}^{(i)}(|\psi\rangle) = r_i\}$, $r_i \in \{0, 1, 2\}$. For example, the rank with respect to qubit 1 divides the space of the pure states of odd n qubits into three families: $F_0^{(1)} = \{|\psi\rangle : \text{rank}^{(1)}(|\psi\rangle) = 0\}$, $F_1^{(1)} = \{|\psi\rangle : \text{rank}^{(1)}(|\psi\rangle) = 1\}$, and $F_2^{(1)} = \{|\psi\rangle : \text{rank}^{(1)}(|\psi\rangle) = 2\}$.

It is not hard to see that two states belong to the same family if and only if they have the same rank with respect to the same qubit. Accordingly, if two states are SLOCC equivalent then they belong to the same family $F_{r_i}^{(i)}$. However, the converse does not hold, i.e., the states in the same family may be inequivalent under SLOCC. It is further noted that the aforementioned three SLOCC families $F_0^{(i)}$, $F_1^{(i)}$ and $F_2^{(i)}$ form a complete partition of the space of odd n qubits. That is, any state of odd n qubits belongs to one and only one of the above three families.

We exemplify the result for the six SLOCC equivalent classes for three qubits: $|GHZ\rangle$, $|W\rangle$, $A-BC$, $B-AC$, $C-AB$ and $A-B-C$ [2]. The rank with respect to qubit i permits the partitioning of the space of three qubits into three families $F_0^{(i)}$, $F_1^{(i)}$ and $F_2^{(i)}$, as illustrated in Table 1.

We also revisit the examples in the last section. Clearly, for any odd $n \geq 5$ qubits, $|GHZ\rangle$ belongs to family $F_2^{(i)}$, the Dicke state $|(n-1)/2, n\rangle$ belongs to family $F_1^{(i)}$, whereas all the full separable states and all the Dicke states $|\ell, n\rangle$ (including n -qubit $|W\rangle$ state) for $1 \leq \ell < (n-1)/2$, belong to family $F_0^{(i)}$, $i = 1, \dots, n$.

3.2 Nine families based on the ranks with respect to qubits 1 and 2

As discussed in the previous section, the rank with respect to qubit 1 divides the space of odd n qubits into three families $F_0^{(1)}$, $F_1^{(1)}$ and $F_2^{(1)}$. For odd $n \geq 5$ qubits, based on the rank with respect to qubit 2 each family $F_{r_1}^{(1)}$, $r_1 \in \{0, 1, 2\}$, can be further divided into three different families: $F_{r_1, r_2}^{(1,2)} = F_{r_1}^{(1)} \cap F_{r_2}^{(2)}$, $r_2 \in \{0, 1, 2\}$. Here, each family $F_{r_1, r_2}^{(1,2)}$ is the intersection of the families $F_{r_1}^{(1)}$ and $F_{r_2}^{(2)}$. More specifically, the family $F_2^{(1)}$ is divided into three families $F_{2,0}^{(1,2)}$, $F_{2,1}^{(1,2)}$ and $F_{2,2}^{(1,2)}$, the family $F_1^{(1)}$ into three families $F_{1,0}^{(1,2)}$, $F_{1,1}^{(1,2)}$ and $F_{1,2}^{(1,2)}$, and the family $F_0^{(1)}$ into three families $F_{0,0}^{(1,2)}$, $F_{0,1}^{(1,2)}$ and $F_{0,2}^{(1,2)}$. For odd $n \geq 5$ qubits, we list the representative states of the families $F_{r_1, r_2}^{(1,2)}$ in Table 2.

Table 1: The three partitions for three qubits

qubit i	family	SLOCC classes
$i = 1$	$F_2^{(1)}$	$ GHZ\rangle$
	$F_1^{(1)}$	$ W\rangle, A - BC$
	$F_0^{(1)}$	$A - B - C, B - AC, C - AB$
$i = 2$	$F_2^{(2)}$	$ GHZ\rangle$
	$F_1^{(2)}$	$ W\rangle, B - AC$
	$F_0^{(2)}$	$A - B - C, A - BC, C - AB$
$i = 3$	$F_2^{(3)}$	$ GHZ\rangle$
	$F_1^{(3)}$	$ W\rangle, C - AB$
	$F_0^{(3)}$	$A - B - C, A - BC, B - AC$

Table 2: The nine families for odd $n \geq 5$ qubits based on the ranks with respect to qubits 1, 2

family	representative state
$F_{2,2}^{(1,2)}$	$ GHZ\rangle$
$F_{2,1}^{(1,2)}$	$\frac{1}{\sqrt{6}}[(0 \cdots 0\rangle + 1 \cdots 1\rangle) + (010 \cdots 010\rangle + 101 \cdots 101\rangle) + (0 \cdots 0110\rangle - 101 \cdots 10001\rangle)]$
$F_{2,0}^{(1,2)}$	$\frac{1}{2}[(0 \cdots 0\rangle + 1 \cdots 1\rangle) + (010 \cdots 010\rangle + 101 \cdots 101\rangle)]$
$F_{1,2}^{(1,2)}$	$\frac{1}{\sqrt{5}}[(001 \cdots 1\rangle - 010 \cdots 0\rangle) + (0110 \cdots 0\rangle + 10 \cdots 0\rangle) + 1101 \cdots 1\rangle]$
$F_{1,1}^{(1,2)}$	$ (n-1)/2, n\rangle$
$F_{1,0}^{(1,2)}$	$\frac{1}{\sqrt{2}}(0 \cdots 0\rangle + 01 \cdots 1\rangle)$
$F_{0,2}^{(1,2)}$	$\frac{1}{2}(0 \cdots 0\rangle + 1 \cdots 1\rangle + 010 \cdots 01\rangle - 101 \cdots 10\rangle)$
$F_{0,1}^{(1,2)}$	$\frac{1}{\sqrt{2}}(0 \cdots 0\rangle + 101 \cdots 1\rangle)$
$F_{0,0}^{(1,2)}$	$ 0 \cdots 0\rangle$

Consequently, the ranks with respect to qubits 1 and 2 divide the space of odd $n \geq 5$ qubits into nine different families. Note furthermore that the nine SLOCC families form a complete partition of the space of odd $n \geq 5$ qubits. That is, any state of odd n qubits belongs to one and only one of the nine families.

Continuing with the example for three qubits, we see that the six SLOCC equivalence classes are divided into five families based on the ranks with respect to qubits 1 and 2, see Table 3.

Table 3: Partition for three qubits based on the ranks with respect to qubits 1, 2

family	SLOCC equivalent class
$F_{2,2}^{(1,2)}$	$ GHZ\rangle$
$F_{1,1}^{(1,2)}$	$ W\rangle$
$F_{1,0}^{(1,2)}$	$A - BC$
$F_{0,1}^{(1,2)}$	$B - AC$
$F_{0,0}^{(1,2)}$	$A - B - C, C - AB$

3.3 3^n families based on the ranks with respect to qubits $1, \dots, n$

Now, assume that the ranks with respect to qubits $1, \dots, (\ell-1)$ permit the partitioning of the space of odd $n \geq 5$ qubits into $3^{(\ell-1)}$ families: $F_{r_1, r_2, \dots, r_{(\ell-1)}}^{(1,2, \dots, (\ell-1))}$, $r_1, r_2, \dots, r_{(\ell-1)} \in \{0, 1, 2\}$. Then, each family $F_{r_1, r_2, \dots, r_{(\ell-1)}}^{(1,2, \dots, (\ell-1))}$ can be further divided into three families: $F_{r_1, r_2, \dots, r_\ell}^{(1,2, \dots, \ell)} = F_{r_1, r_2, \dots, r_{(\ell-1)}}^{(1,2, \dots, (\ell-1))} \cap F_{r_\ell}^{(\ell)}$, $r_\ell \in \{0, 1, 2\}$

based on the rank with respect to qubit ℓ . Clearly, each family $F_{r_1, r_2, \dots, r_\ell}^{(1, 2, \dots, \ell)}$ is associated with the sequence $\{r_1, \dots, r_\ell\}$, $r_1, \dots, r_\ell \in \{0, 1, 2\}$, and different sequences correspond to different families. Consequently, in total there are 3^ℓ SLOCC families based on the ranks with respect to qubits $1, \dots, \ell$. In particular, there are 3^n SLOCC families based on the ranks with respect to qubits $1, \dots, n$. It should be noted that at least one family contains an infinite number of SLOCC classes.

It is readily seen that n -qubit $|GHZ\rangle$ state belongs to family $F_{2, \dots, 2}^{(1, \dots, n)}$, the Dicke states $|(n-1)/2, n\rangle$ and $|(n+1)/2, n\rangle$ belong to family $F_{1, \dots, 1}^{(1, \dots, n)}$, whereas all the full separable states and all the Dicke states $|\ell, n\rangle$ (including n -qubit $|W\rangle$ state), with $1 \leq \ell < (n-1)/2$ and $n \geq 5$, belong to family $F_{0, \dots, 0}^{(1, \dots, n)}$. It is worth pointing out that all the states in the family $F_{2, \dots, 2}^{(1, \dots, n)}$ are genuinely entangled as discussed in section 2.

For any state $|\psi\rangle$ of odd n qubits, by computing $rank^{(i)}(|\psi\rangle)$, $i = 1, \dots, \ell$ ($\leq n$), we can determine which family the state $|\psi\rangle$ belongs to. It is plain to see that two states belong to the same family if and only if they have the same ranks with respect to qubits $1, \dots, \ell$ ($\leq n$). Thus, if two states are SLOCC equivalent then they belong to the same family.

Consider once again the example for three qubits. A straightforward calculation demonstrates that the six SLOCC equivalence classes of three qubits are divided into six families based on the ranks with respect to qubits 1, 2 and 3, i.e., each family is just a single SLOCC class, see Table 4.

Table 4: Partition for three qubits based on the ranks with respect to qubits 1, 2, 3

family	SLOCC equivalent class
$F_{2, 2, 2}^{(1, 2, 3)}$	$ GHZ\rangle$
$F_{1, 1, 1}^{(1, 2, 3)}$	$ W\rangle$
$F_{1, 0, 0}^{(1, 2, 3)}$	$A - BC$
$F_{0, 1, 0}^{(1, 2, 3)}$	$B - AC$
$F_{0, 0, 1}^{(1, 2, 3)}$	$C - AB$
$F_{0, 0, 0}^{(1, 2, 3)}$	$A - B - C$

4 Discussion

In [22], the “filter” approach was used to separate SLOCC orbits and it was shown that the following four five-qubit states

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|11111\rangle + |00000\rangle), \quad (4.1)$$

$$|\Phi_2\rangle = \frac{1}{2}(|11111\rangle + |11100\rangle + |00010\rangle + |00001\rangle), \quad (4.2)$$

$$|\Phi_3\rangle = \frac{1}{\sqrt{6}}(\sqrt{2}|11111\rangle + |11000\rangle + |00100\rangle + |00010\rangle + |00001\rangle), \quad (4.3)$$

$$|\Phi_4\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|11111\rangle + |10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle) \quad (4.4)$$

are in different orbits.

We now classify the above four states using our framework. Based on the ranks with respect to qubits 1, 2 and 3, $|\Phi_1\rangle$ belongs to $F_{2, 2, 2}^{(1, 2, 3)}$, $|\Phi_2\rangle$ belongs to $F_{0, 0, 0}^{(1, 2, 3)}$, $|\Phi_3\rangle$ belongs to $F_{0, 0, 1}^{(1, 2, 3)}$, and $|\Phi_4\rangle$ belongs to $F_{1, 1, 1}^{(1, 2, 3)}$, in agreement with [22] that the four states are in different orbits.

Also note that the space of five qubits is divided into nine different families based on the ranks with respect to qubits 1 and 2. We list the representatives of the nine families in Table 5.

Table 5: The nine families for five qubits based on the ranks with respect to qubits 1, 2

family	representative state
$F_{2,2}^{(1,2)}$	$ GHZ\rangle$
$F_{2,1}^{(1,2)}$	$\frac{1}{\sqrt{6}}(00000\rangle + 11111\rangle + 01010\rangle + 10101\rangle + 00110\rangle - 10001\rangle)$
$F_{2,0}^{(1,2)}$	$\frac{1}{2}(00000\rangle + 11111\rangle + 01010\rangle + 10101\rangle)$
$F_{1,2}^{(1,2)}$	$\frac{1}{\sqrt{5}}(00111\rangle - 01000\rangle + 01100\rangle + 10000\rangle + 11011\rangle)$
$F_{1,1}^{(1,2)}$	$ 2, 5\rangle$
$F_{1,0}^{(1,2)}$	$\frac{1}{\sqrt{2}}(00000\rangle + 01111\rangle)$
$F_{0,2}^{(1,2)}$	$\frac{1}{2}(00000\rangle + 11111\rangle + 01001\rangle - 10110\rangle)$
$F_{0,1}^{(1,2)}$	$\frac{1}{\sqrt{2}}(00000\rangle + 10111\rangle)$
$F_{0,0}^{(1,2)}$	$ 00000\rangle$

5 Conclusion

In this paper, we have introduced the rank with respect to qubit i for any state of odd n qubits and established its invariance under SLOCC. That is, if two states are SLOCC equivalent then they have the same ranks with respect to qubits $1, \dots, n$. The ranks with respect to qubits $1, \dots, n$ permit the partitioning of the space of odd $n \geq 5$ qubits into 3^n inequivalent families. It is straightforward to know that two states belong to the same family if and only if they have the same ranks with respect to qubits $1, \dots, n$. In other words, all the states of a family have the same ranks with respect to qubits $1, \dots, n$. As a consequence, if two states are SLOCC equivalent then they belong to the same family. Furthermore, each family corresponds to the sequence $\{r_1, \dots, r_n\}$, $r_i \in \{0, 1, 2\}$, and different families correspond to different sequences. In terms of the ranks, we have given a necessary condition for a pure state to be biseparable and a sufficient condition for a pure state to be genuinely entangled. The classification based on the ranks of states may possess more physical meaning. As a final note, we would like to mention that the SLOCC invariance of the rank for odd n qubits does not hold for even n qubits.

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Appendix

We here give the proof of Eq. (2.10). We distinguish two cases: $i = 1$ and $2 \leq i \leq n$.

Case 1. $i = 1$.

In this case, Eq. (2.10) becomes

$$M(|\psi\rangle) = \mathcal{A}_1 M(|\psi'\rangle) \mathcal{A}_1^T \det(\mathcal{A}_2) \cdots \det(\mathcal{A}_n). \quad (5.1)$$

Let $|\psi\rangle$ and $|\psi'\rangle$ be related by Eq. (2.11), and $\mathcal{A}_1 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$. It is easy to verify that Eq. (5.1) holds if and only if the following three SLOCC equations hold together:

$$\begin{aligned} T(|\psi\rangle) &= [P(|\psi'\rangle)\alpha_1\alpha_3 + T(|\psi'\rangle)(\alpha_2\alpha_3 + \alpha_1\alpha_4) + Q(|\psi'\rangle)\alpha_2\alpha_4] \\ &\quad \times \det(\mathcal{A}_2) \cdots \det(\mathcal{A}_n), \end{aligned} \quad (5.2)$$

$$\begin{aligned} P(|\psi\rangle) &= [P(|\psi'\rangle)\alpha_1^2 + 2T(|\psi'\rangle)\alpha_1\alpha_2 + Q(|\psi'\rangle)\alpha_2^2] \\ &\quad \times \det(\mathcal{A}_2) \cdots \det(\mathcal{A}_n), \end{aligned} \quad (5.3)$$

$$\begin{aligned} Q(|\psi\rangle) &= [P(|\psi'\rangle)\alpha_3^2 + 2T(|\psi'\rangle)\alpha_3\alpha_4 + Q(|\psi'\rangle)\alpha_4^2] \\ &\quad \times \det(\mathcal{A}_2) \cdots \det(\mathcal{A}_n). \end{aligned} \quad (5.4)$$

Notice that $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n$ can be written as $(\mathcal{A}_1 \otimes \mathcal{I}_2 \otimes \cdots \otimes \mathcal{I}_n) \circ (\mathcal{I}_1 \otimes \mathcal{A}_2 \otimes \mathcal{I}_3 \otimes \cdots \otimes \mathcal{I}_n) \circ \cdots \circ (\mathcal{I}_1 \otimes \cdots \otimes \mathcal{I}_{n-1} \otimes \mathcal{A}_n)$, then Eqs. (5.2), (5.3) and (5.4) follow immediately from the two lemmas below.

Lemma 1. For odd n qubits, if $|\psi\rangle$ and $|\psi'\rangle$ are related by

$$|\psi\rangle = \underbrace{\mathcal{A}_1 \otimes \mathcal{I}_2 \otimes \cdots \otimes \mathcal{I}_n}_n |\psi'\rangle, \quad (5.5)$$

then

$$T(|\psi\rangle) = P(|\psi'\rangle)\alpha_1\alpha_3 + T(|\psi'\rangle)(\alpha_2\alpha_3 + \alpha_1\alpha_4) + Q(|\psi'\rangle)\alpha_2\alpha_4, \quad (5.6)$$

$$P(|\psi\rangle) = P(|\psi'\rangle)\alpha_1^2 + 2T(|\psi'\rangle)\alpha_1\alpha_2 + Q(|\psi'\rangle)\alpha_2^2, \quad (5.7)$$

$$Q(|\psi\rangle) = P(|\psi'\rangle)\alpha_3^2 + 2T(|\psi'\rangle)\alpha_3\alpha_4 + Q(|\psi'\rangle)\alpha_4^2. \quad (5.8)$$

Proof. We only prove Eq. (5.6). The proofs for Eqs. (5.7) and (5.8) are analogous. By Eq. (5.5), we obtain

$$a_i = \alpha_1 b_i + \alpha_2 b_{2^{n-1}+i}, \quad a_{2^{n-1}+i} = \alpha_3 b_i + \alpha_4 b_{2^{n-1}+i}, \quad (5.9)$$

for $0 \leq i \leq 2^{n-1} - 1$. By substituting Eq. (5.9) into Eq. (2.2), we obtain

$$T(|\psi\rangle) = \sum_{i=0}^{2^{n-1}-1} (-1)^{N(i)} (\alpha_1 b_i + \alpha_2 b_{2^{n-1}+i}) (\alpha_3 b_{2^{n-1}-i-1} + \alpha_4 b_{2^n-i-1}). \quad (5.10)$$

Note that $T(|\psi'\rangle)$, $P(|\psi'\rangle)$, and $Q(|\psi'\rangle)$ can be rewritten as:

$$T(|\psi'\rangle) = \sum_{i=0}^{2^{n-1}-1} (-1)^{N(i)} b_{2^{n-1}+i} b_{2^{n-1}-i-1}, \quad (5.11)$$

$$P(|\psi'\rangle) = \sum_{i=0}^{2^{n-1}-1} (-1)^{N(i)} b_i b_{2^{n-1}-i-1}, \quad (5.12)$$

$$Q(|\psi'\rangle) = \sum_{i=0}^{2^{n-1}-1} (-1)^{N(i)} b_{2^{n-1}+i} b_{2^n-i-1}. \quad (5.13)$$

Expanding Eq. (5.10) and using Eqs. (5.11), (5.12) and (5.13) yield Eq. (5.6).

Lemma 2. For odd n qubits, if $|\psi\rangle$ and $|\psi'\rangle$ are related by

$$|\psi\rangle = \underbrace{\mathcal{I}_1 \otimes \cdots \otimes \mathcal{I}_{k-1} \otimes \mathcal{A}_k \otimes \mathcal{I}_{k+1} \otimes \cdots \otimes \mathcal{I}_n}_n |\psi'\rangle, \quad (5.14)$$

then

$$T(|\psi\rangle) = T(|\psi'\rangle) \det(\mathcal{A}_k), \quad (5.15)$$

$$P(|\psi\rangle) = P(|\psi'\rangle) \det(\mathcal{A}_k), \quad (5.16)$$

$$Q(|\psi\rangle) = Q(|\psi'\rangle) \det(\mathcal{A}_k), \quad (5.17)$$

for $2 \leq k \leq n$.

Proof. We only prove Eq. (5.15). The proofs for Eqs. (5.16) and (5.17) can be given analogously. It is sufficient to consider $k = 2$. Let $\mathcal{A}_2 = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$. Then, by Eq. (5.14), we obtain

$$a_i = \beta_1 b_i + \beta_2 b_{2^{n-2}+i}, \quad (5.18)$$

$$a_{2^{n-2}+i} = \beta_3 b_i + \beta_4 b_{2^{n-2}+i}, \quad (5.19)$$

$$a_{2^{n-1}+i} = \beta_1 b_{2^{n-1}+i} + \beta_2 b_{2^{n-1}+2^{n-2}+i}, \quad (5.20)$$

$$a_{2^{n-1}+2^{n-2}+i} = \beta_3 b_{2^{n-1}+i} + \beta_4 b_{2^{n-1}+2^{n-2}+i}, \quad (5.21)$$

for $0 \leq i \leq 2^{n-2} - 1$. We may rewrite $T(|\psi\rangle)$ in Eq. (2.2) as

$$\begin{aligned}
T(|\psi\rangle) &= \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} a_i a_{2^n-i-1} \\
&\quad - \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} a_{2^{n-2}+i} a_{2^{n-1}+2^{n-2}-i-1} \\
&\quad - \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} a_{2^{n-1}+i} a_{2^{n-1}-i-1} \\
&\quad + \sum_{i=0}^{2^{n-2}-1} (-1)^{N(i)} a_{2^{n-1}+2^{n-2}+i} a_{2^{n-2}-i-1}.
\end{aligned} \tag{5.22}$$

Substituting Eqs. (5.18), (5.19), (5.20) and (5.21) into Eq. (5.22) yields the desired result Eq. (5.15).

Case 2. $2 \leq i \leq n$.

We give a brief proof here. After a tedious but straightforward calculation, the following identity holds:

$$(1, i) \circ (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n) \circ (1, i) = \mathcal{A}_i \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{i-1} \otimes \mathcal{A}_1 \otimes \mathcal{A}_{i+1} \otimes \cdots \otimes \mathcal{A}_n. \tag{5.23}$$

Letting $M^{(i)} = M \circ (1, i)$ and using Eq. (2.11), we have

$$\begin{aligned}
M^{(i)}(|\psi\rangle) &= M^{(i)}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n |\psi'\rangle) \\
&= M \circ (1, i) \circ (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n) \circ (1, i) ((1, i) |\psi'\rangle).
\end{aligned} \tag{5.24}$$

By substituting Eq. (5.23) into Eq. (5.24), then using Eq. (5.1), we obtain that

$$M^{(i)}(|\psi\rangle) = \mathcal{A}_i M((1, i) |\psi'\rangle) \mathcal{A}_i^T \det(\mathcal{A}_1) \cdots \det(\mathcal{A}_{i-1}) \det(\mathcal{A}_{i+1}) \cdots \det(\mathcal{A}_n), \tag{5.25}$$

and then Eq. (2.10) follows immediately.

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